Optimal Rate Allocation for Shape-Gain Gaussian Quantizers

ISIT 2001

June 26, 2001

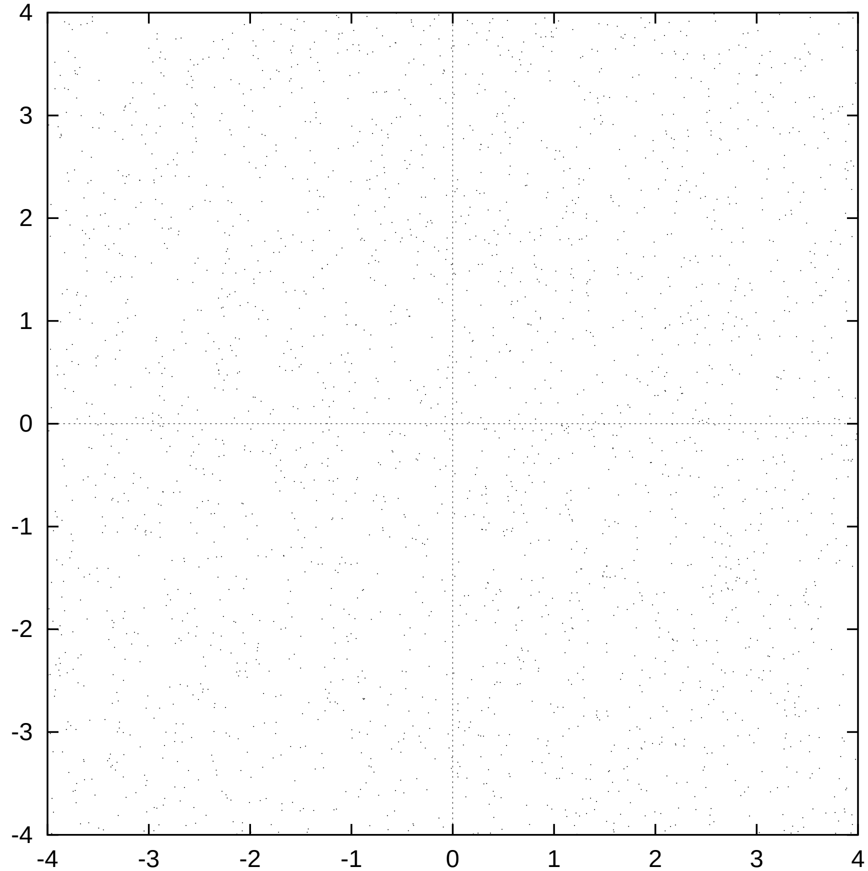
Jon Hamkins
Jet Propulsion Laboratory

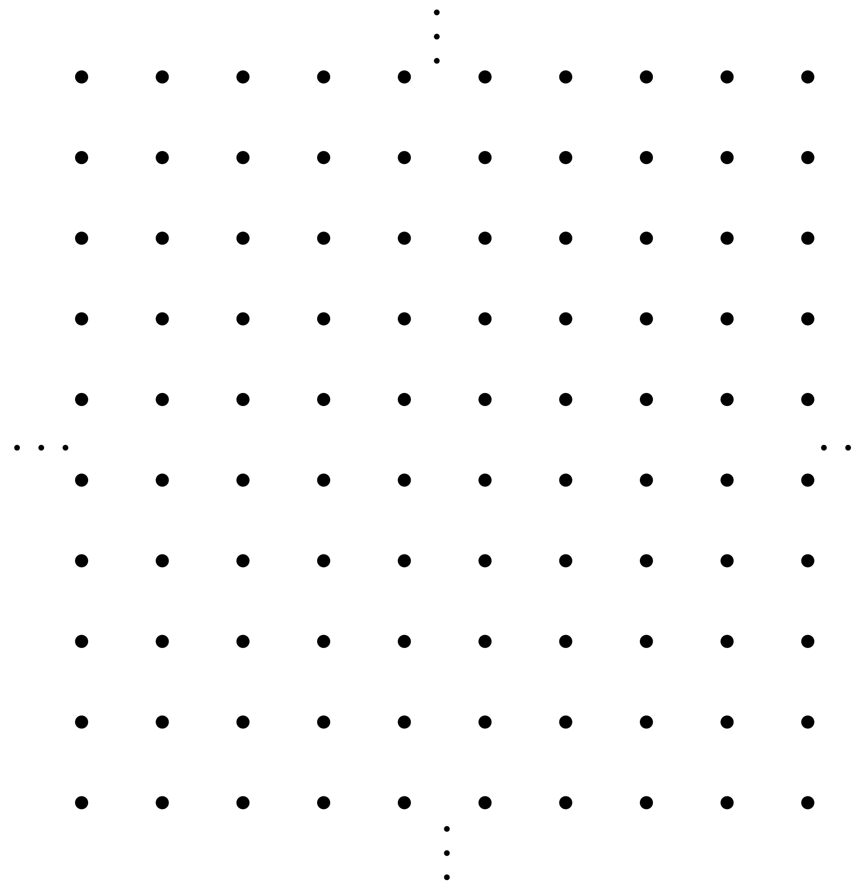
Ken Zeger
University of California, San Diego

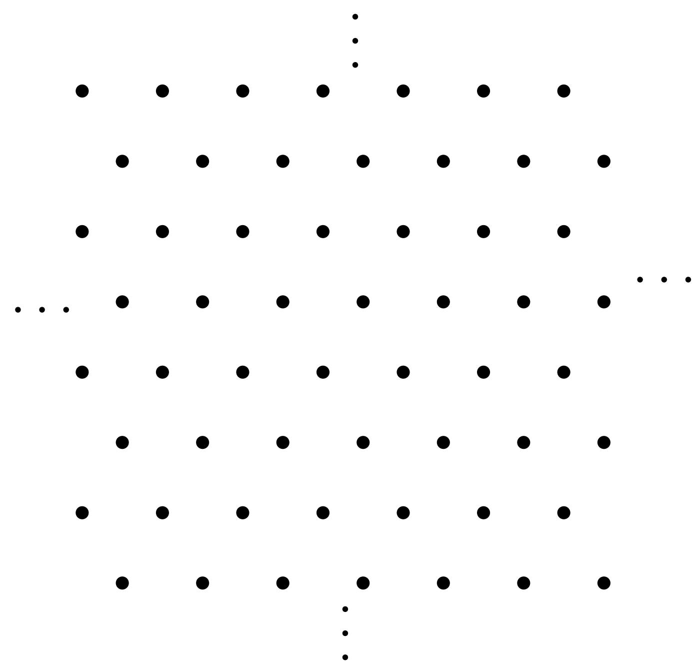
Introduction – Vector Quantization

Group source samples into a vector, and quantize the whole vector.

Example: i.i.d. Uniform source, formed into 2D vectors







Quantization Coefficient

The quantization coefficient: average mean squared error per dimension for high rate quantization of a uniform source (scaled so as to be a dimensionless quantity).

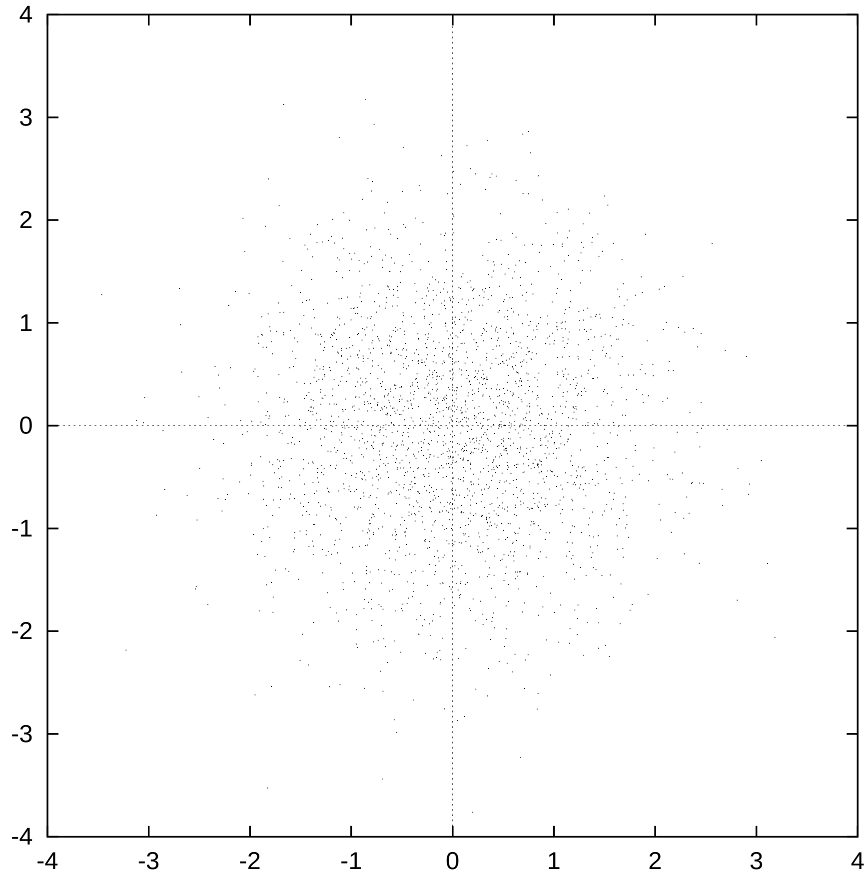
Lattice VQ's are good for uniform sources.

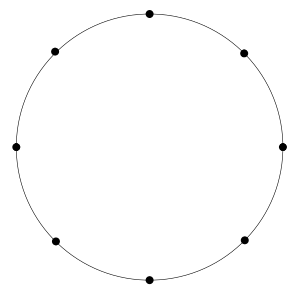
Dimension	Lattice	Quantization coefficient
k	$\mathbb{Z}^k$	$\frac{1}{12} \approx 0.0833$
2	A_2	$\frac{5}{36\sqrt{3}} \approx 0.0802$
3	A_3	0.0787
3	A_3^*	0.0785
24	Λ_{24}	0.0658
$\to \infty$		optimal: $\frac{1}{2\pi e} \approx 0.0586$

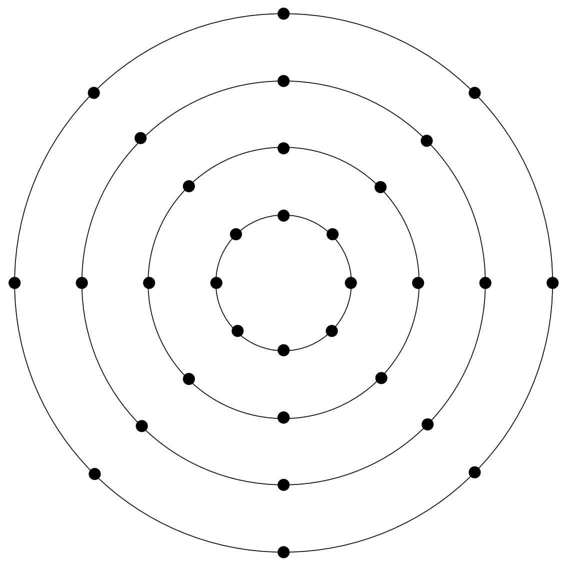
A Memoryless Gaussian Source

Let $X = (X_1, \dots, X_k)$, where $X_i \sim N(0, \sigma^2)$.

In two dimensions ($k = 2$, $\sigma^2 = 1$):







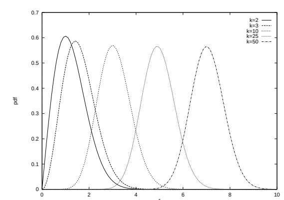
Properties of a Memoryless Gaussian Source

If $X = (x_1, \ldots, x_k)$, $x_i \sim N(0, \sigma^2)$, and if $Y = (y_1, \ldots, y_k)$, then the prob. dist'n. has spherical symmetry:

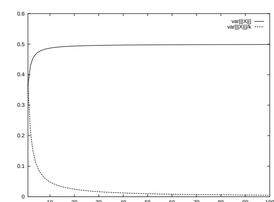
$$f_X(Y) = \prod_{i=1}^{k} \frac{\exp\left(\frac{-y_i^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}} = \frac{\exp\left(\frac{-\|Y\|^2}{2\sigma^2}\right)}{(2\pi\sigma^2)^{k/2}}$$

Properties of the gain $\|X\|$:

pdf: $$f_{\|X\|}(r) = \frac{2r^{k-1}\exp\left(\frac{-r^2}{2\sigma^2}\right)}{\Gamma(k/2)(2\sigma^2)^{k/2}}$$



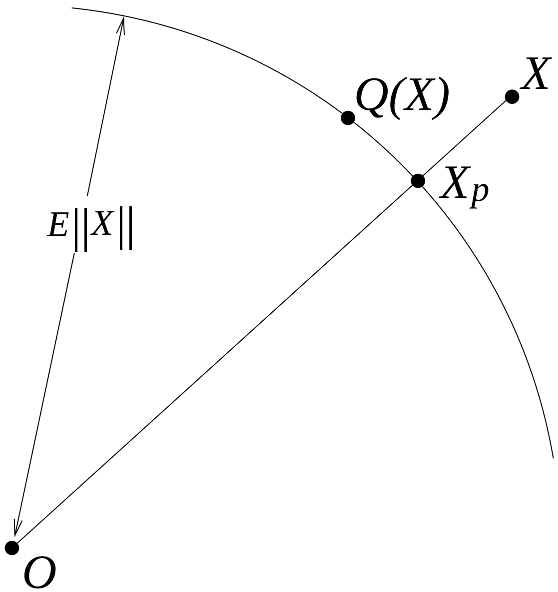
mean: $$E[\|X\|] = \frac{\sqrt{2\sigma^2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{\sqrt{2\pi\sigma^2}}{\beta\left(\frac{k}{2}, \frac{1}{2}\right)} \approx \sigma\sqrt{k - (1/2)}$$



variance: $$\mathrm{var}[\|X\|] = k\sigma^2 - \frac{2\pi\sigma^2}{\beta^2\left(\frac{k}{2}, \frac{1}{2}\right)} \approx \frac{\sigma^2}{2}$$

Thus, $\|X\|/\sqrt{k\sigma^2}$ has tends to unit-mean, zero-variance as $k \to \infty$. This is "sphere-hardening".

Motivation for Spherical Vector Quantization

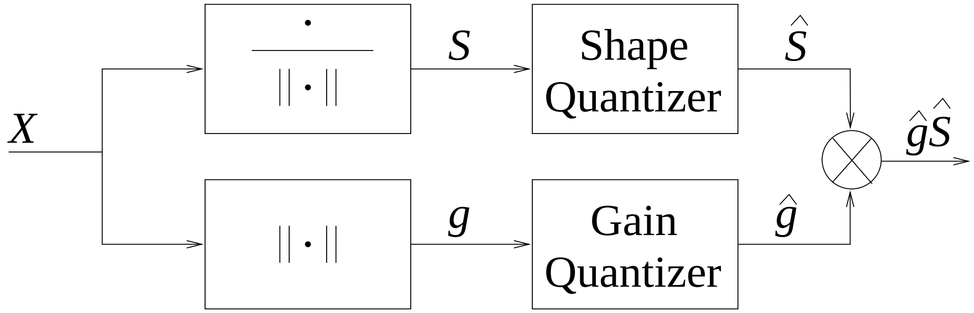


Sakrison showed

$$
\begin{aligned}
D &= \frac{1}{k} E\|X - Q(X)\|^2 \\
&= \frac{1}{k} E\|X_p - Q(X)\|^2 + \frac{1}{k} \underbrace{E\|X - X_p\|^2}_{\text{var}\|X\|} \\
&= D_s + D_g \\
&\approx D_s, \text{ for large } k
\end{aligned}
$$

Wrapped Spherical Vector Quantizer

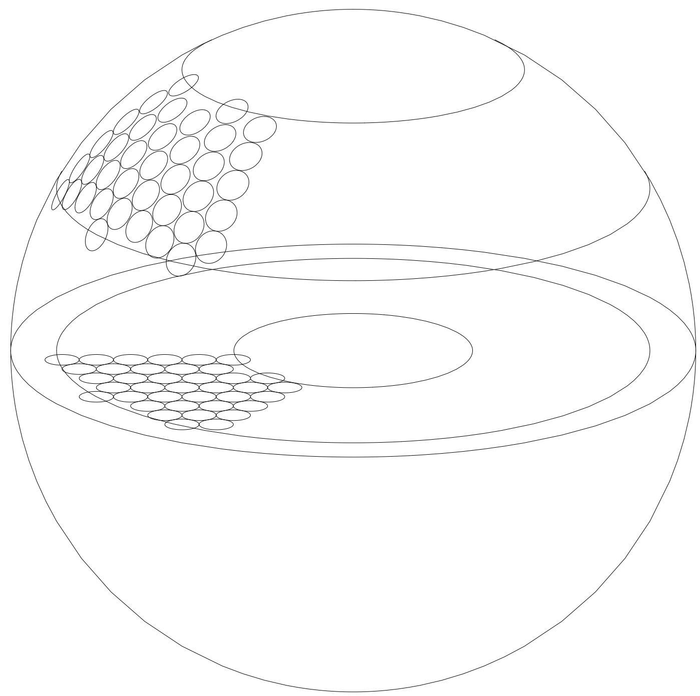
Shape-gain approach: $g = \|X\|$, and $S = \frac{X}{g}$



Gain codebook: scalar quantizer optimized by Lloyd-Max algorithm.

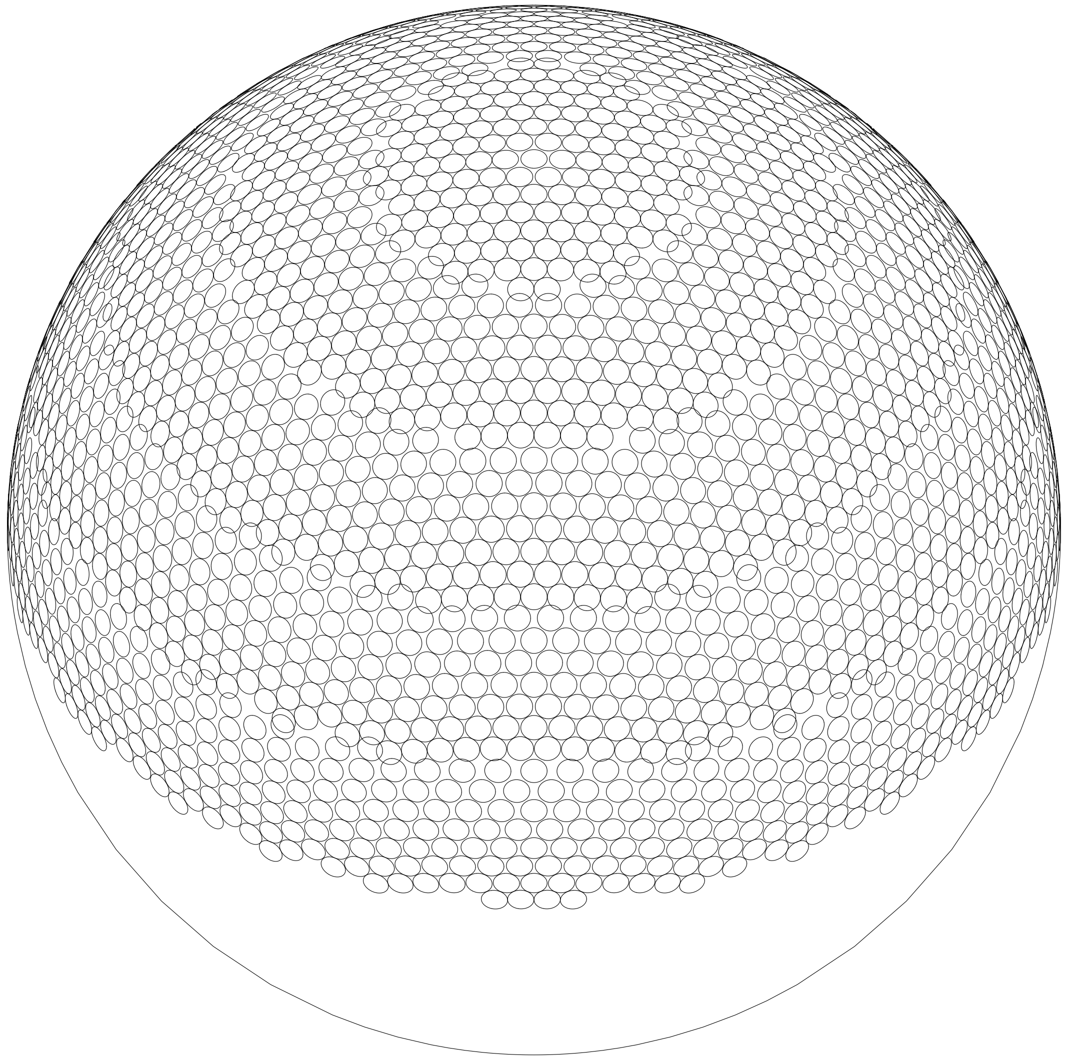
Wrapped spherical code construction

Within each annulus, introduce only small distortion:

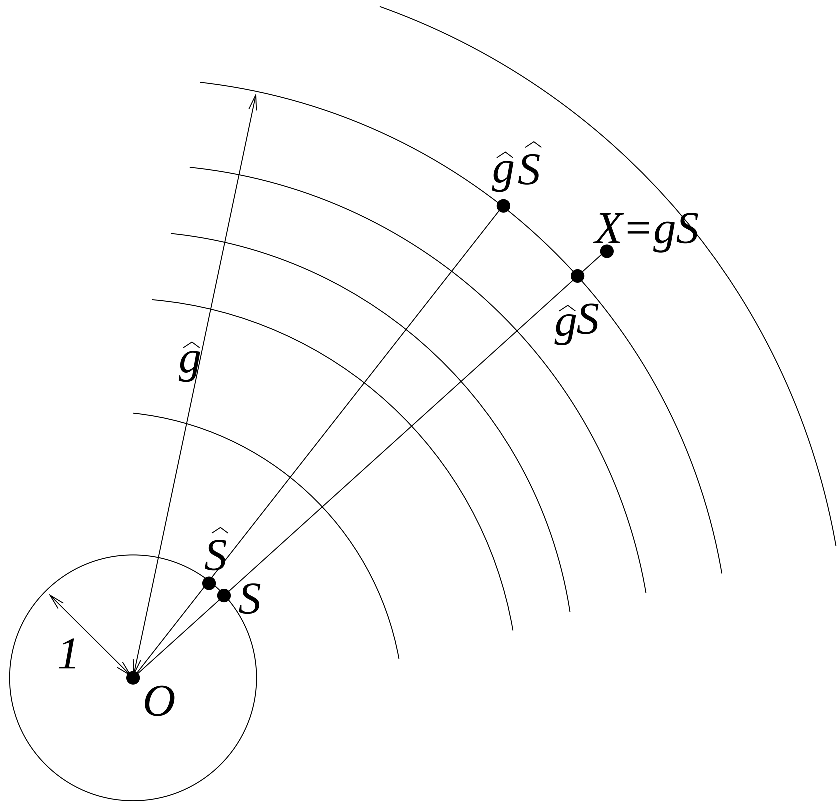


Theorem: The quantization coefficient of a wrapped spherical code is within $O(\sqrt{d})$ of the quantization coefficient of the underlying packing used to construct it.

Example of shape codebook



Performance Analysis



$$
\begin{aligned}
D &= \frac{1}{k} E\|X - \hat{g}\hat{S}\|^2 \\
&= D_g + D_s \\
&= \frac{1}{k} E(g - \hat{g})^2 + \frac{1}{k} E\hat{g}^2 E\|S - \hat{S}\|^2
\end{aligned}
$$

Performance Analysis

In order to optimize rate allocation, $D = D_g + D_s$ must be estimated under differing shape and gain codebook sizes.

Finite rate:

Evaluate $D_g = \frac{1}{k} E(g - \hat{g})^2$ using $f_g(r)$ and table of $\hat{g}$ outputs.

Evaluate $D_s = \frac{1}{k} \underbrace{E\hat{g}^2}_{1} \underbrace{E\|S - \hat{S}\|^2}_{2}$ where

1. $E\hat{g}^2 = Eg^2 - E(g - \hat{g})^2 \approx Eg^2 = \sigma^2$
2. $$\begin{aligned} E\|S - \hat{S}\|^2 &\approx \left(\begin{array}{l}\text{MSE of underlying lattice } \Lambda \text{ used as } (k-1)\text{-dimensional quantizer for} \\ \text{uniform source.}\end{array}\right) \\ &= (k-1)\sigma^2 G(\Lambda) V(\Lambda)^{\frac{2}{k-1}} \end{aligned}$$

Asymptotic rate:

Use Bennett's integral to write $D_g \approx C_g 2^{-2R_g k} = C_g 2^{-2k(R - R_s)}$

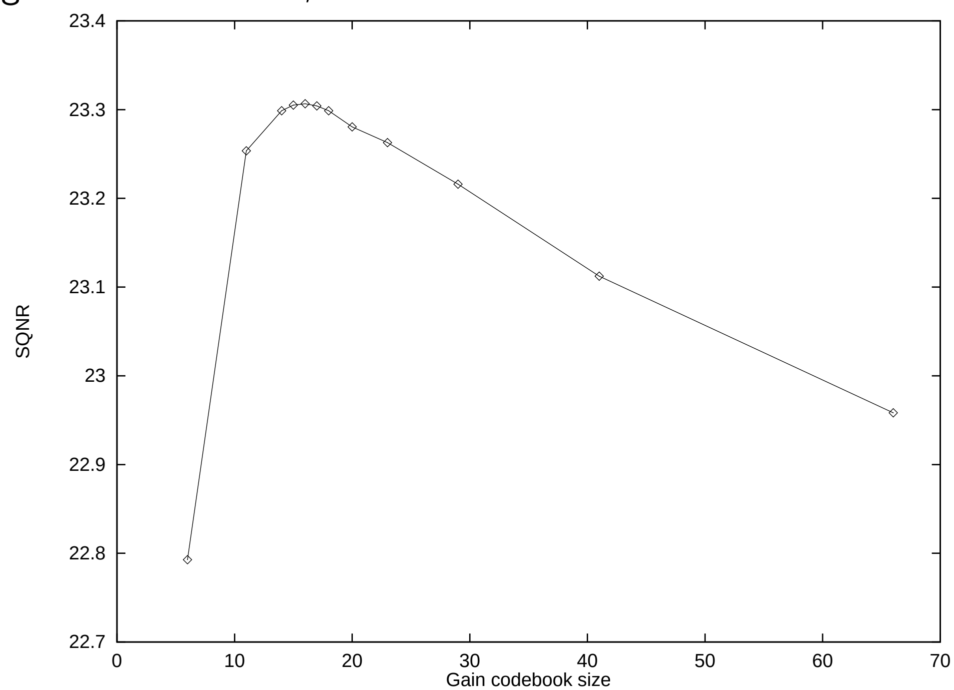
Write $D_s \approx (k-1)\sigma^2 G(\Lambda) V(\Lambda)^{\frac{2}{k-1}} \approx C_s 2^{-2R_s\left(\frac{k}{k-1}\right)}$

C_g and C_s are constants independent of R_s and R_g.

Shape-Gain Rate Allocation

$$\begin{aligned} R &= \frac{1}{k}\log_2\left[(\text{Gain CB size}) \times (\text{Shape CB size})\right] \\ &= \frac{1}{k}\log_2(\text{Gain CB size}) + \frac{1}{k}\log_2(\text{Shape CB size}) \\ &= R_g + R_s \end{aligned}$$

SQNR as a function of gain codebook size, $R = 2$



Optimize rate allocation by maximizing SQNR using numerical methods.

Shape-Gain Rate Allocation – Asymptotically High Rate

Theorem Let $X \in \mathbb{R}^k$ be an uncorrelated Gaussian vector with zero mean and component variances σ^2 and let Λ be a lattice in $\mathbb{R}^{k-1}$ with normalized second moment $G(\Lambda)$. Suppose X is quantized by a k-dimensional shape-gain vector quantizer at rate $R = R_s + R_g$ (where R_s and R_g are the shape and gain quantizer rates) with independent shape and gain encoders and whose shape codebook is a wrapped spherical code constructed from Λ. Then as $R \to \infty$, the minimum mean squared quantization error D decays as

$$D \approx C_s \left(\frac{k}{k-1}\right) \left(\frac{C_g}{C_s}(k-1)\right)^{1/k} \cdot 2^{-2R} \tag{1}$$

and is achieved by

$$\begin{aligned} R_s &= \left(\frac{k-1}{k}\right) \left[R + \frac{1}{2k} \log_2 \left(\frac{C_s}{C_g} \cdot \frac{1}{k-1}\right)\right] \\ R_g &= \left(\frac{1}{k}\right) \left[R - \frac{k-1}{2k} \log_2 \left(\frac{C_s}{C_g} \cdot \frac{1}{k-1}\right)\right] \end{aligned}$$

where $C_s = \sigma^2 \cdot (k-1)G(\Lambda) \left(\frac{2\pi^{k/2}}{\Gamma(k/2)}\right)^{\frac{2}{k-1}}$ and $C_g = \sigma^2 \cdot \frac{3^{k/2}\Gamma^3\left(\frac{k+2}{6}\right)}{8\Gamma(k/2)}$.

Comments and Conclusions

Comments:

- For large R, $R_s \approx \left(\frac{k-1}{k}\right) R$ and $R_g \approx \frac{1}{k} R$. This means that the shape codebook should have about $2^{(k-1)R}$ codevectors and the gain codebook should have about 2^R scalar codepoints, as intuition would indicate.
- The asymptotic formula is also quite accurate for moderate R. In simulations of the wrapped Leech lattice spherical vector quantizer, we observed that the optimal gain codebook rate was within 8% of this figure when $R \geq 3$ and within 1% when $R \geq 5$.

Conclusions:

- Performance and complexity of quantizers for memoriless Gaussian sources is equivalent to the performance and complexity of quantizers for the uniform source.
- Optimal rate allocations for finite rate and asymptotically large rate were determined.